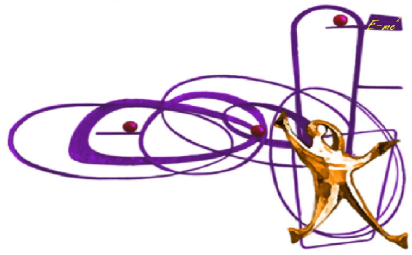


SECOND YEAR ACTIVITY REPORT

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My research activity during the second year has been focused on two main topics: quantum control and quantum resource theory. In this report I will present an overview of the results obtained, along with mathematical and numerical techniques employed. Since the work I will present is a natural continuation of my first year activity, I will mention – if needed – previously obtained results, summarizing them as briefly as possible.

1 Quantum Control and Product Formulas

Given a quantum system whose state evolves in its Hilbert space $\mathscr{H}$, we are interested in procedures which can be used to control the free evolution of the system $U(t) = \mathrm{e}^{-itH}$, i.e. the evolution generated by its Hamiltonian H, in order to avoid some unwanted detrimental effects, such as decoherence due to the interaction with the environment. In many situation the unwanted effects can be eliminated by constraining the system to evolve in some subspaces $\mathscr{H}_\mu$ of the total Hilbert space $\mathscr{H}$, in the sense that it should never make transitions between different subspaces, if we want to avoid unwanted interactions with the environment. In such cases controlling the system amounts to the realization of an experimental procedure which guarantees a partitioning of the Hilbert space $\mathscr{H}$ into subspaces $\mathscr{H}_\mu$ among which transitions are inhibited:

$$\mathscr{H} = \bigoplus_\mu \mathscr{H}_\mu \tag{1}$$

However, it is of vital importance for practical applications (quantum information processing, etc.) that the invasiveness of the control procedure should be contained to the minimum required, meaning that inside each subspace the system should evolve as if the controls were not acting. A procedure satisfying these requirements should result into an effective Hamiltonian for the system in the form

$$H_Z = \sum_\mu P_\mu H P_\mu, \tag{2}$$

where P_μ is the projection onto the subspace $\mathscr{H}_\mu$, i.e: $P_\mu |\psi\rangle \in \mathscr{H}_\mu$ for every $|\psi\rangle \in \mathscr{H}$. Starting from techniques studied during the first year,[1] we found a generalized control protocol[2] which extends previous known results. It consists in the alternation of the free evolution of the system, according to its own Hamiltonian H, for a time t/n, with an evolution under the action of the control potential $V = \sum_\mu \lambda_\mu P_\mu$, whose strength can be controlled by a tunable parameter K:

$$U_{n,K}(t) = \left(\mathrm{e}^{-\mathrm{i}\frac{t}{n}KV}\mathrm{e}^{-\mathrm{i}\frac{t}{n}H}\right)^n. \tag{3}$$

An equivalent physical implementation of this evolution consists in the alternation of the two Hamiltonians H and V for time intervals lasting respectively t/n and Kt/n, as shown pictorially in Figure 1

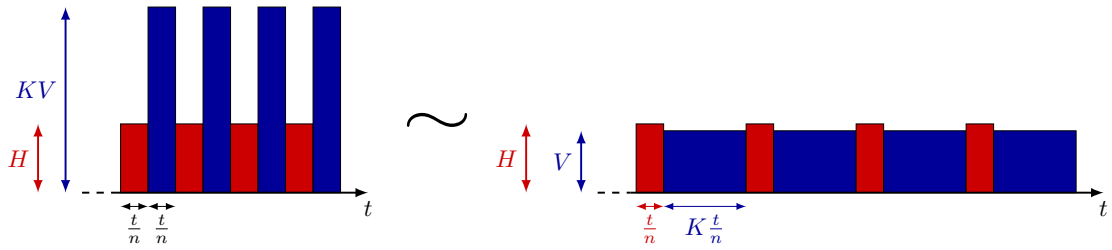


Figure 1: Two equivalent physical pictures giving the evolution (3).

By considering $K = \mathcal{O}(n)$, we can obtain as a particular case the pulsed procedure, which yields the controlled dynamics $U_Z(t) = \mathrm{e}^{-\mathrm{i}tH_Z}$ in the limit of very frequent pulses, i.e. $n \to \infty$. We extended this result by showing that the requirement $K(n) = \mathcal{O}(n)$ is not necessary in order to achieve a controlled evolution: in particular, requiring only that $K(n) \to \infty$ as $n \to \infty$ (but not necessarily as fast as $K(n) = \mathcal{O}(n)$), we managed to show that[2]

$$U_{n,K(n)}(t) = \mathrm{e}^{-\mathrm{i}tK(n)V}e^{-\mathrm{i}tH_Z} + \mathcal{O}\left(\frac{1}{K(n)}\right) \tag{4}$$

This result paves the way to more efficient quantum control techniques, since in standard techniques the total time of the control pulses (in Figure 1 the total base length of blue rectangles on the right) grows as $K(n) = \mathcal{O}(n)$; our result shows that it is possible to achieve a controlled dynamics also using shorter kicks, say $K(n) \sim n^\alpha$, with $0 \leqslant \alpha \leqslant 1$.

2 Quantum resource theory

The theoretical and experimental development of quantum information protocols has motivated the study of "manipulations" of quantum states using classes of operations restricted by some physical constraints. These constraints define a subset of all the operations which in principle could be performed on the quantum systems. States which can be prepared within this set of *allowed operations* are called *free states*, while all the states which cannot be prepared satisfying the restrictions are called *resource states*. The name comes from the fact that when a resource state is made available, it can be used to perform some task which is outside the set of allowed operations (with the state being consumed in the process, or being converted into a less resourceful state). Although resource states cannot be created using only the *allowed operations*, we can use allowed operations to convert some resource state into another.

2.1 LOCC operations and Majorization

When considering the operations that two (or more) *distant* parties A and B can perform, the physical constraint corresponds to locality: A and B can only control their systems with local unitaries, perform local measurements and they can freely communicate classical data with each other (which includes the results of their local measurements). The corresponding class of allowed operations is known in the literature as *local operations assisted by classical communication* (LOCC). In the resource theory of LOCC operations separable states are free, while entangled ones are resource states. An important question to be answered in this resource theory is then when some particular state of a bipartite system (i.e. composed of two *distant* parties) can be converted into another. Considering state conversions between pure states of bipartite systems, i.e. whose Hilbert space has the tensor product structure $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, we say that $|\psi\rangle$ can be LOCC-converted into $|\varphi\rangle$ and we write $|\psi\rangle \to |\varphi\rangle$ if there exists a LOCC operation T such that $T(|\psi\rangle\langle\psi|) = |\varphi\rangle\langle\varphi|$, while we say that the conversion is possible with probability of success at most p and we write $|\psi\rangle \xrightarrow{p} |\varphi\rangle$ if

$$\langle\varphi| T(|\psi\rangle\langle\psi|) |\varphi\rangle \leqslant p \tag{5}$$

for all LOCC operations T; equation (5) means that a measurement performed on any state $T(|\psi\rangle\langle\psi|)$ – reachable from $|\psi\rangle$ through a LOCC operation T – cannot collapse in the state $|\varphi\rangle$ with a probability higher than p. A complete characterization in terms of local properties is known for conversions between pure states of a bipartite system.[3,4] Specifically, given a state $|\psi\rangle$ in $\mathscr{H}_A \otimes \mathscr{H}_B$, the local state on one of the subsystems – e.g. A – is given by the density matrix $\psi_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$, where tr_B denotes the partial trace over $\mathscr{H}_B$, and we denote with $\lambda(\psi_A)$ the list of eigenvalues of ψ_A. We call $\lambda(\psi_A)$ the "local spectrum" of $|\psi\rangle$ (it does not depend on which system we trace out, i.e. $\lambda(\psi_A) = \lambda(\psi_B)$). Local spectra of two states $|\psi\rangle$ and $|\varphi\rangle$ determine completely if they can be deterministically converted into each other, or if a stochastic conversion with success probability p is possible. More precisely, defining $n = \dim \mathscr{H}_A$ and $m = \dim \mathscr{H}_B$, and taking for definiteness (without loss of generality) $n \leqslant m$, the maximal success probability in a conversion from $|\psi\rangle$ to $|\varphi\rangle$ is given by[5]

$$\max_{T \in \mathrm{LOCC}} \langle\varphi| T(|\psi\rangle\langle\psi|) |\varphi\rangle = \min_{1 \leqslant l \leqslant n} \frac{\sum_{k=l}^{n} \lambda(\psi_A)_k^{\downarrow}}{\sum_{k=l}^{n} \lambda(\varphi_A)_k^{\downarrow}} \equiv \Pi(\lambda(\psi_A), \lambda(\varphi_A)). \tag{6}$$

2.1.1 The problem

A long standing conjecture[3] asserts that if two states $|\psi\rangle$ and $|\varphi\rangle$ are picked at random from $\mathscr{H}_A \otimes \mathscr{H}_B$, the probability that it is possible to (deterministically) LOCC-convert one state into another goes to zero as $n \to \infty$, i.e. $P(\Pi(\lambda(\psi_A), \lambda(\varphi_A)) = 1) \to 0$. We analysed numerically such conjecture, using numerical techniques which enabled us to investigate the high-dimensional limit, and extending the study also to non-deterministic conversions, studying the distribution of the maximal success probability $\Pi(\lambda(\psi_A), \lambda(\varphi_A))$ as $|\psi\rangle$ and $|\varphi\rangle$ are sampled randomly according to the unitarily invariant measure on $\mathscr{H}_A \otimes \mathscr{H}_B$.

2.1.2 Numerical methods

The numerical sampling of states in $\mathscr{H}_A \otimes \mathscr{H}_B$ requires the storage of nm complex numbers, and an ordinary algorithm leading to the local spectrum of interest would require an amount of $\mathcal{O}(mn^2)$ complex operations (and recalling that $m \geqslant n$, this number is

$\mathcal{O}(n^3)$ in the best case). Such scaling in the computational complexity of the algorithms is a strong limitation in our investigation, since we are interested in the high-dimensional limit $n, m \to \infty$. For this reason we used some results on random matrix theory which allowed us to sample the states local spectra (the pairs $\lambda(\psi_A), \lambda(\varphi_A)$) without actually sampling the states.[5] The method consists in the sampling of sparse $n \times n$ matrices (instead of $n \times m$ dense ones) with a tridiagonal structure.[6,7] In this tridiagonal model the largest dimension m appears only as a parameter of the distribution used to sample the diagonal and sub-diagonal elements. This translates in a reduction of the storage needed from $\mathcal{O}(mn^2)$ to simply $\mathcal{O}(n)$, and a reduction in the computational complexity from $\mathcal{O}(mn^2)$ to $\mathcal{O}(n^2)$ (which comes from the tridiagonal structure). Since the computational complexity does not scale with m anymore, we can investigate, for each fixed choice of n, arbitrarily high values of m.

2.1.3 Results

- For fixed n, numerical computations show that the sequence $P(|\psi\rangle \to |\varphi\rangle)$ is non-decreasing in m. Therefore, as $m \to \infty$, the probability of local conversion converges to a positive constant:

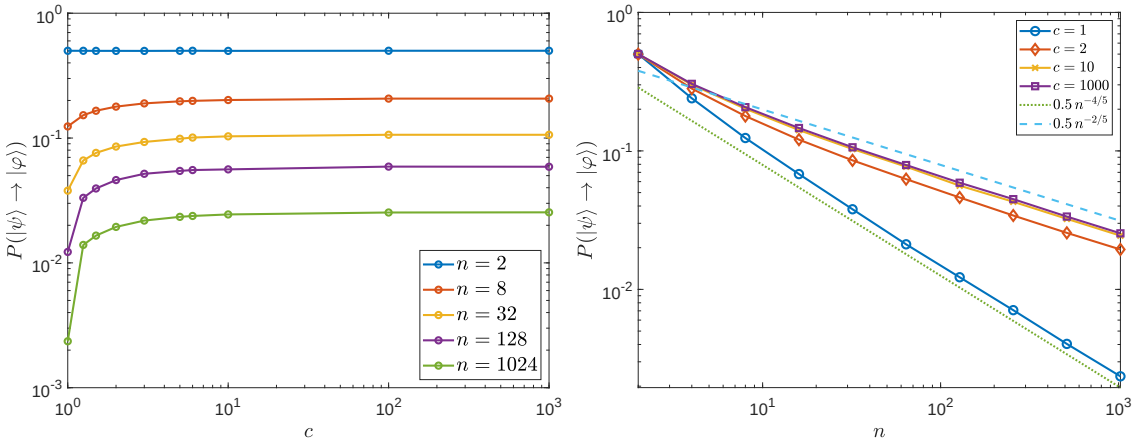
$$\lim_{m\to\infty} P(|\psi\rangle \to |\varphi\rangle) = \kappa(n) > 0. \tag{7}$$

  The sequence of constants $\kappa(n)$ is decreasing in n. See Figure 2(a).

- We verified numerically that when both $n, m \to \infty$ the relative volume of pairs of LOCC-convertible states goes to zero:

$$\lim_{n,m\to\infty} P(|\psi\rangle \to |\varphi\rangle) = 0. \tag{8}$$

  (This is a precise statement of Nielsen's conjecture.) See Figure 2(b).



(a) (b)

Figure 2: The probability $P(|\psi\rangle \to |\varphi\rangle)$ for random pure states ψ and φ versus the parameter c for several values of n (a), and versus the dimension n for several values of the parameter $c = m/n$ (b).

- We can analyse the asymptotic rate at which the probability of local conversion decays to zero. More precisely, we consider the limit $n, m \to \infty$ with the ratio $\frac{m}{n} = c$ fixed. We are assuming $c \geqslant 1$, but this is not a limitation since the problem is symmetric under the transformation $c \leftrightarrow c^{-1}$ (and the fixed point of this transformation

plays indeed a special role as explained later). The analysis of the problem reveals a mapping to the calculation of the persistence probability for a non-Markovian random walk. This connection, combined with numerical evidence (see Figure 2), suggests that the probability of local conversion decays *algebraically* at large n. Indeed, we find numerically that

$$P\left(|\psi\rangle \to |\varphi\rangle\right) = P\left(\max_{T\in\mathrm{LOCC}} \langle\varphi|\, T(|\psi\rangle\langle\psi|)\, |\varphi\rangle = 1\right) \simeq \frac{b}{n^\theta}, \quad \text{as } n\to\infty \text{ with } m = cn. \tag{9}$$

We find persistence exponents $\theta \simeq 4/5$ if $c = 1$, and $\theta \simeq 2/5$ if $c > 1$ (see Figure 2(b)).

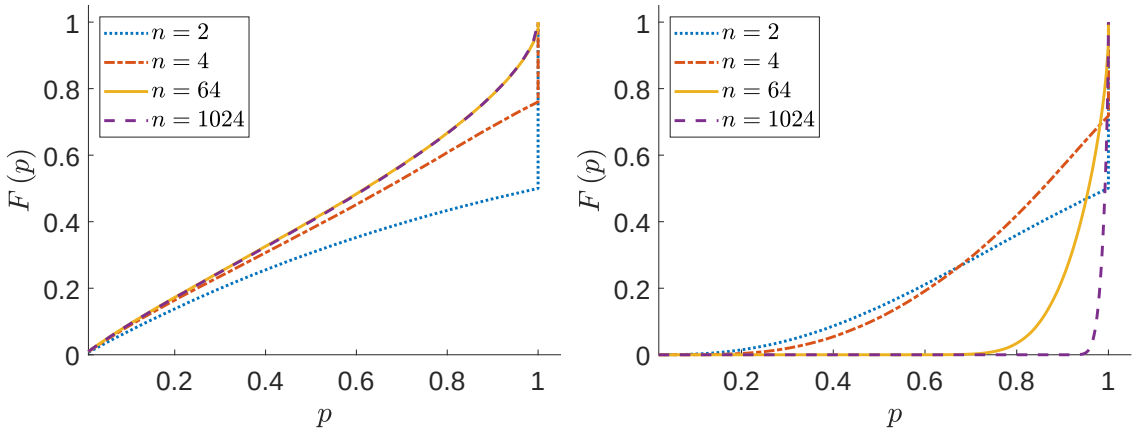
- We computed numerically the probability distribution

$$F(p) = P\left(\max_{T\in\mathrm{LOCC}} \langle\varphi|\, T(|\psi\rangle\langle\psi|)\, |\varphi\rangle\right) \tag{10}$$

for several values of n and m. The cumulative distribution $F(p)$ has a jump at $p = 1$ (see Figure 3), hence the probability density $f(p) = F'(p)$ has a continuous part (supported on the interval $0 \le p \le 1$) and a singular part at $p = 1$:

$$f(p) = f_{\mathrm{cont}}(p) + \kappa\delta(p-1). \tag{11}$$

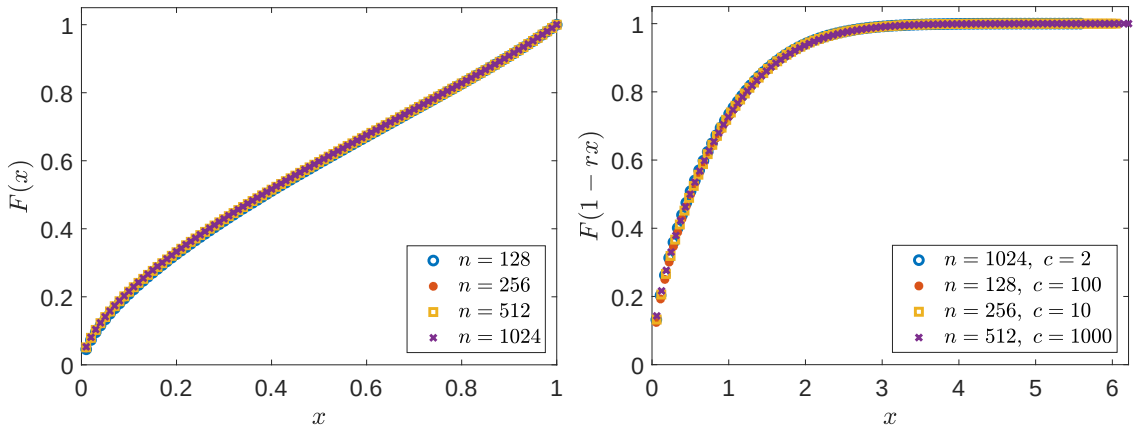
The height of the jump $F(1) - F(1^-) = \kappa = P\left(\max_{T\in\mathrm{LOCC}} \langle\varphi|\, T(|\psi\rangle\langle\psi|)\, |\varphi\rangle = 1\right)$ vanishes in the limit $n \to \infty$ with $m = cn$. While the weight of singular part at $p = 1$ vanishes for large n, the continuous part of the density for $c > 1$ concentrates around (on the left of) $p = 1$.



(a) $c = 1$ (b) $c = 2$

Figure 3: Distribution function $F(p)$ of $\max_{T\in\mathrm{LOCC}} \langle\varphi|\, T(|\psi\rangle\langle\psi|)\, |\varphi\rangle$. The decomposition (11) of the corresponding probability density function $f(p)$ into a singular and a continuous component can be seen from the presence of a jump at $p = 1$. In both the balanced and unbalanced cases the height of the jumps vanishes as n grows.

- It can be noted that the cases $c = 1$ and $c > 1$ are different, especially in the continuous part $f_{\mathrm{cont}}(p)$ of the distribution, which in the unbalanced case $(c > 1)$ shows a concentration around $p = 1$ (see Figure 3(b)), while in the balanced case $(c = 1)$ maintains its support over the whole interval $0 \leqslant p \leqslant 1$. The physical consequence of this behaviour is that while the probability of obtaining a deterministic conversion $P\left(\max_{T\in\mathrm{LOCC}} \langle\varphi|\, T(|\psi\rangle\langle\psi|)\, |\varphi\rangle = 1\right)$ conversion goes to zero as n increases, the



(a) (b)

Figure 4: Universal scaling of the distribution of $\max_{T \in \mathrm{LOCC}} \mathrm{Tr}\,(\varphi T \psi)$. See Eqs. (13)–(14). Panel (a) shows the balanced case ($c = 1$), while panel (b) shows the unbalanced case ($c > 1$) and $r = c^{1/6}(\sqrt{c} - 1)^{2/3} n^{2/3}$.

maximal success rate $p = \langle\varphi| T(|\psi\rangle\langle\psi|) |\varphi\rangle$ in a stochastic conversion gets arbitrarily close to one in the same limit, in the case $c > 1$.

- We found that the different behaviour of the two cases $c = 1$ and $c > 1$ can be understood in terms of relative fluctuations of the minimum eigenvalue $\lambda_n(\psi_A)$, which is a quantity of interest in the field of random matrix theory. Thanks to this connection we also were able to find a rescaling of the variable $\langle\varphi| T(|\psi\rangle\langle\psi|) |\varphi\rangle$ leading to a universal law in distribution for large n and m. In particular, for $c > 1$ fixed, we find that (see Figure 4(b))

$$\lim_{n\to\infty} P\left(\max_{T \in \mathrm{LOCC}} \langle\varphi| T(|\psi\rangle\langle\psi|) |\varphi\rangle > 1 - \frac{x}{c^{1/6} \left|1 - \sqrt{c}\right|^{2/3} n^{2/3}} \right) = H_{\mathrm{unb}}(x), \tag{12}$$

for some nontrivial scaling functions $H_{\mathrm{unb}}(x)$. We can equivalently write for the probability distribution

$$\lim_{n\to\infty} F\left(1 - \frac{x}{c^{1/6} \left|1 - \sqrt{c}\right|^{2/3} n^{2/3}} \right) = 1 - H_{\mathrm{unb}}(x). \tag{13}$$

For $c = 1$ (the fixed point of the symmetry $c \leftrightarrow c^{-1}$) we observe instead a simpler asymptotics

$$\lim_{n\to\infty} F(1 - x) = 1 - H_{\mathrm{bal}}(x). \tag{14}$$

See Figure 4(a).

2.2 Precision enhancement in Quantum Metrology with Gaussian states

Quantum resources, such as entanglement and coherence, can be used to perform high sensitive measurements, surpassing the precision which is achievable by using only classical resources.

The maximal precision achievable in some measurement is ultimately determined by the discrete nature of all kind of measurements (from the discrete nature of the electric charge in electronic devices, to the particle nature of light in optical devices), which gives rise to a Poissonian noise. This noise strongly limits the precision achievable in

the measurement with the so-called shot-noise factor $1/\sqrt{N}$, where N is the number of "probes" (e.g. electrons, photons) employed in the experiment. However, using states of the probes with genuinely quantum features such as entanglement or coherence, one can improve this precision to a factor $1/N$, known as Heisenberg limit.[8]

However, the quantum states needed to reach this performance enhancement can be difficult to create and manipulate experimentally, reducing the usefulness of such precision enhancement. Gaussian quantum states provide a promising avenue from this point of view, since they are easy to create and manipulate experimentally. However, it is not yet clear under what conditions Gaussian states alone can be used to obtain an Heisenberg scaling precision enhancement, although some advancements have recently been made in this direction.[9]

In this framework, we have considered a parameter estimation problem in which only Gaussian states and Gaussian measurements (measurements whose results have a Gaussian probability distribution when carried out on Gaussian states) can be performed.[10] We have shown that within this experimentally realistic restrictions, defining a *Gaussian resource theory*,[11] it is still possible to achieve an Heisenberg scaling precision.

We considered the problem of estimating a generic parameter φ encoded into an M-port linear passive interferometer. The parameter can be any feature characterizing the interferometer, e.g. an optical phase, the reflectivity of a beam splitter, or even the result of some combination between different parameters pertaining to separate components of the interferometer. We use an input state $\hat{\rho}$ to "probe" the interferometer, whose action on the state is described by some unitary $\hat{U}_\varphi$ through $\hat{\rho}_\varphi = \hat{U}_\varphi \hat{\rho} \hat{U}_\varphi^\dagger$. The information about the parameter φ is thus encoded into an output state $\hat{\rho}_\varphi$, so that the parameter can be estimated from the results of a measurement performed on $\hat{\rho}_\varphi$, with POVM elements $\{\hat{\Pi}_x\}$. By Born rule, the statistics of measurement results is described by $p(x|\varphi) = \mathrm{tr}(\hat{\rho}_\varphi \hat{\Pi}_x)$. This statistics depends then both on the measurement being performed, i.e. $\{\hat{\Pi}_x\}$, and on the output state $\hat{\rho}_\varphi$, which in turn depends on the input state ρ and on the structure of the interferometer.

It is known from classical estimation theory that the minimum error achievable in the estimation of a parameter φ is given by the Cramér-Rao bound:

$$\delta\varphi \geq \delta\varphi_{\min} = \frac{1}{\sqrt{\nu F(\varphi)}}, \tag{15}$$

where ν is the number of measurements performed and $F(\varphi)$ is the Fisher Information (FI), defined as:

$$F(\varphi) = \int p(x|\varphi) \left(\frac{\partial \log p(x|\varphi)}{\partial \varphi} \right)^2 \mathrm{d}x. \tag{16}$$

Since $p(x|\varphi)$ is determined by $\hat{\rho}$, $\hat{U_\varphi}$ and $\hat{\Pi}_x$, the maximization of $F(\varphi)$ for a fixed interferometer requires the combined optimization of both the input state and the measurement being performed.

If only classical resources are employed, $F(\varphi)$ cannot grow faster than linearly in the injected number of photons N, yielding shot-noise limited precision, scaling as $\delta\varphi \sim 1/\sqrt{N}$, while quantum protocols can raise this scaling factor to N^2, yielding a precision which scales as $\delta\varphi \sim 1/N$. We provided a measurement scheme achieving the Heisenberg scaling if some reasonable physically motivated conditions are satisfied.

As input state in our scheme we considered the single mode-squeezed vacuum

$$|\psi_0\rangle = \hat{V}_{\mathrm{in}} \hat{S}_1(r_0) |\Omega\rangle, \qquad |\Omega\rangle = |0\rangle^{\otimes M}, \tag{17}$$

where $\hat{V}_{\text{in}}$ is a passive unitary which can be chosen arbitrarily, while $\hat{S}_j(r_0)$ is the squeezing operator of the $j-$th mode:

$$\hat{S}_j(r_0) = e^{\frac{1}{2}r_0\left(\hat{a}_j^2 - \hat{a}_j^{\dagger 2}\right)}, \tag{18}$$

and the squeezing parameter r_0 determines the average number of particles, according to the relation:

$$N = \langle\psi_0|\,\hat{N}\,|\psi_0\rangle = \sinh^2(r_0) \tag{19}$$

We then considered an homodyne measurement (the paradigmatic Gaussian measurement in quantum optics) at the first output mode, which is described by the POVM elements $\hat{\Pi}_x$:

$$\hat{\Pi}_x = \hat{V}_{\text{out}}^{\dagger} e^{i\vartheta\hat{a}_1^{\dagger}\hat{a}_1}\,|x\rangle_{11}\langle x|\,e^{-i\vartheta\hat{a}_1^{\dagger}\hat{a}_1}\hat{V}_{\text{out}}. \tag{20}$$

This measurement scheme is pictorially represented in Figure 5

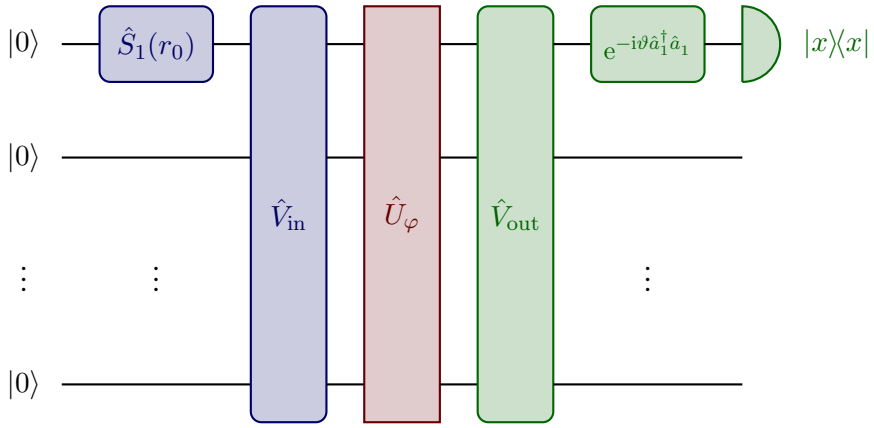


Figure 5

In this measurement scheme we have found[10] that it is possible to achieve an Heisenberg scaling if the following conditions are satisfied:

- The unitaries $\hat{V}_{\text{in}}$ and $\hat{V}_{\text{out}}$, representing respectively the input and the output preparatory stages, must satisfy the condition:

  $$|(V_{\text{out}}U_{\varphi}V_{\text{in}})_{11}|^2 = 1 - \frac{c}{N}, \qquad c \geqslant 0, \tag{21}$$

  which corresponds to the physical condition that (almost) all the photons which are sent to the first input port end up in the first output port of the circuit[1], where the measurement is being carried out.

- The local oscillator phase ϑ in the homodyne detection must be chosen according to the condition:

  $$\vartheta = \arg[(V_{\text{out}}U_{\varphi}V_{\text{in}})_{11}] \pm \arctan\left(e^{2r_0}\right), \tag{22}$$

  which tells us the particular quadrature of the electromagnetic field to measure in order to achieve the most sensible measurement to variations of the parameter φ.

[1]More precisely, if $c = 0$ this condition assures that all the photons employed in the experiment are efficiently used to probe the circuit (no energy is wasted). However, we prove that it is not strictly necessary that $|(V_{\text{out}}U_{\varphi}V_{\text{in}})_{11}|^2 = 1$, so that we can obtain Heisenberg scaling sensitivity also if there is a certain probability of losing photons (as long as it is of order $1/N$)

In order to satisfy the above requirement (21), an adaptive procedure should be performed, aimed at realizing one of the two (mathematically equivalent, but physically different) conditions:

$$(V_{\rm out})_{1i} = (V_{\rm in}^\dagger U_\varphi^\dagger)_{1i} + \mathcal{O}\left(\frac{1}{N}\right), \quad \text{or} \quad (V_{\rm in})_{i1} = (U_\varphi^\dagger V_{\rm out}^\dagger)_{i1} + \mathcal{O}\left(\frac{1}{N}\right) \tag{23}$$

which correspond respectively to fixing the input stage and optimizing the output stage accordingly, or vice versa, fixing the output stage optimizing the input. We proved that this adaptive procedure can be carried out only on one side (which is practically more feasible). If for example we keep fixed the (non-optimized) input preparatory stage while optimizing the input, we asymptotically obtain the following Fisher information:

$$F(\varphi) \sim 8(V_{\rm in}^\dagger G_\varphi V_{\rm in})_{11}^2 N(N+1), \tag{24}$$

where

$$G_\varphi = {\rm i} U_\varphi^\dagger \frac{\partial U_\varphi}{\partial \varphi} \tag{25}$$

represents the generator of the circuit. Equation (24) shows that $F(\varphi)$ scales as N^2 with a prefactor which depends on the non-optimized unitary stage $V_{\rm in}$. The determination of the average and the typical value of this prefactor $(V_{\rm in}^\dagger G_\varphi V_{\rm in})_{11}^2$ is essential from a physical point of view, since small values of such factor can drastically invalidate the Heisenberg scaling $F(\varphi) \sim N^2$ if N is large but finite. For this purpose, we used techniques of random matrix theory to compute the average value of the Fisher information (24) over random choices of the non-optimised unitary stage $V_{\rm in}$ according to the Haar measure over the $M \times M$ unitary matrices, obtaining the result:

$$\mathbb{E}[F(\varphi)] = 8\frac{{\rm Tr}\big(G_\varphi^2\big) + {\rm Tr}(G_\varphi)^2}{M(M+1)} N(N+1), \tag{26}$$

which gives us some criteria on which codifications of the parameter φ in the circuit U_φ are good on average, in terms of the generator G_φ. However, good results on average are not sufficient to assure good results most of the times, since the average of a random variable does not necessarily represent its typical behaviour. Using concentration of measure in high dimensional probability spaces we managed to prove that the average behaviour does indeed represent the typical behaviour if M is large enough, since we are able to prove that

$$P(|F - \mathbb{E}[F]| \geqslant \varepsilon) \leqslant 2{\rm e}^{-\frac{2M\varepsilon^2}{CL^2}}, \tag{27}$$

where $C > 0$ is constant and $L = 32\,\|G\|_\varphi^2\, N(N+1)$. This result implies that the probability of getting results far from the average is exponentially supressed in the number of the interferometer ports M, which tells us that a multi-port interferometer can be an advantageous choice in some cases.

Conferences

During my second year I have attended the following scientific conferences:

- "Current Problems in Theoretical Physics", XXV edition, April 15^{th}-16^{th}, 2019, *Vietri sul Mare* (Italy).
- "51^{st} Symposium on Mathematical Physics", June 16^{th}-18^{th}, 2019, *Toruń* (Poland) with **poster presentation** "Generalized product formulas and quantum control".
- "12^{th} Italian Quantum Information Science Conference", September, 9^{th}-12^{th}, 2019, *Milan* (Italy)
  with **poster presentation** "Optimal Quantum Metrology with Squeezed states".
- "5^{th} International Conference for Young Quantum Information Scientists", September 25^{th}-27^{th}, 2019

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