First year activity report

Giovanni Gramegna
Dottorato di ricerca in Fisica, XXXIII ciclo
Dipartimento Interateneo di Fisica "M. Merlin"

Research activity

My research activity during the first year has been focused on two main topics: Quantum Control and Resource Theories in Thermodynamics.

In regard to quantum control, we have found that it is possible to achieve a controlled dynamics using two different techniques, one hinging upon the application of fast, strong pulses to the system to be controlled, and the other consisting in a strong coupling with a control potential. We have seen that the limiting dynamics is the same in the two cases, and starting from this equivalence we have also analysed the validity of a generalised Trotter product formula.

During my first year I have also investigated issues of interest in the framework of resource theories. A resource theory is defined by restricting the set of operations which can be performed on a system, and aims at determining which state conversions can be achieved given such restrictions. The first systematic example of resource theory in physics is the resource theory of entanglement.[1] However, this approach is not limited to quantum mechanics and one can recognise in it the paradigm which is followed in classical thermodynamics. For this reason there has been a recent surge of interest[2–4] towards the attempt to cast thermodynamics as a resource theory, in order to characterise the behaviour of thermodynamic laws also at the microscopic level, when quantum effects become relevant.[5] In a given resource theory it is possible to establish a partial order over the set of states which can be expressed through a relation called majorization. My research activity in this context has been focused on understanding the statistical behaviour of majorization when considering high dimensional Hilbert spaces, corresponding to the thermodynamic limit.

Continuous and Pulsed Quantum Control

I have studied two techniques which can be used in order to steer the evolution of a quantum system in some desired fashion. In our approach, given a quantum system described by a Hamiltonian H, which generates the evolution of the system in the Hilbert space $\mathscr{H}$, we obtain a controlled dynamics if we are able to act on the system in such a way that the Hilbert space is partitioned into dynamical superselection sectors among which transitions are hindered during the evolution. At the level of the dynamical description, this translates into an effective Hamiltonian of the form:

$$H_Z = \sum_\mu P_\mu H P_\mu, \tag{1}$$

where P_μ are the projections onto the dynamically generated superselection sectors $\mathscr{H}_\mu = P_\mu \mathscr{H}$ of the total Hilbert space $\mathscr{H}$. From the very structure of the Hamiltonian (1) we see that it cannot carry a system which is initially in a certain sector $\mathscr{H}_\mu$ to a different sector $\mathscr{H}_\nu$, and in this sense we have realized a partitioning of the Hilbert space into superselection sectors:

$$\mathscr{H} = \bigoplus_\mu \mathscr{H}_\mu. \tag{2}$$

The first technique that I have considered recalls what is known in the literature as bang-bang control,[6] consisting in the periodic application of fast, unitary pulses to the system under control. In the context of dynamical decoupling it is commonly believed that in order to being able to decouple the system from all its interactions with the environment one needs a sequence of pulses which must close a group of unitaries. In our work[7] we show that this is not necessary, and that it is possible to control the evolution of a quantum system using just a single unitary pulse U_{kick} periodically repeated during the evolution. More precisely, if we denote with P_μ the eigenprojections of the kicks:

$$U_{\text{kick}} = \sum_\mu e^{i\lambda_\mu} P_\mu, \tag{3}$$

and apply n kicks periodically in the time interval $(0, t)$, we show that in the large-n limit the evolution approaches the controlled dynamics generated by the Hamiltonian (1):

$$\left(U_{\text{kick}} e^{-i\frac{t}{n}H}\right)^n = U_{\text{kick}}^n e^{-itH_Z} + \mathcal{O}\left(\frac{1}{n}\right) \tag{4}$$

The limiting evolution one obtains using dynamical decoupling techniques has exactly the same features as a Quantum Zeno Dynamics, which arises when frequent non-selective measurements are performed on a system in order to ascertain if it is in some subspaces of the total Hilbert space. Recalling that a measurement is effected by coupling the system with an external potential describing the measurement apparatus, it has been shown that this same dynamics can also be obtained with a strong continuous coupling with an external potential V, which implies that it is possible to control the quantum system by suitably engineering this external potential. This second technique that I have studied in my work can be expressed as:

$$e^{-it(H+KV)} = e^{-itKV} e^{-itH_Z} + \mathcal{O}\left(\frac{1}{K}\right) \tag{5}$$

valid in the limit of large coupling constant K. Here the superselection sectors are the eigenspaces of the control potential: $V = \sum_\mu \lambda_\mu P_\mu$.

Since both techniques lead (in their respective limits) to the same dynamics, I have compared them through some product formulae in order to better understand this equivalence. From a numerical analysis it turns out that it is possible to slightly generalize the celebrated Trotter product formula to the following one:

$$\left(e^{-i\frac{t}{n}K(n)V} e^{-i\frac{t}{n}H}\right)^n = e^{-it(K(n)V+H)} + \mathcal{O}\left(\frac{1}{n}\right), \tag{6}$$

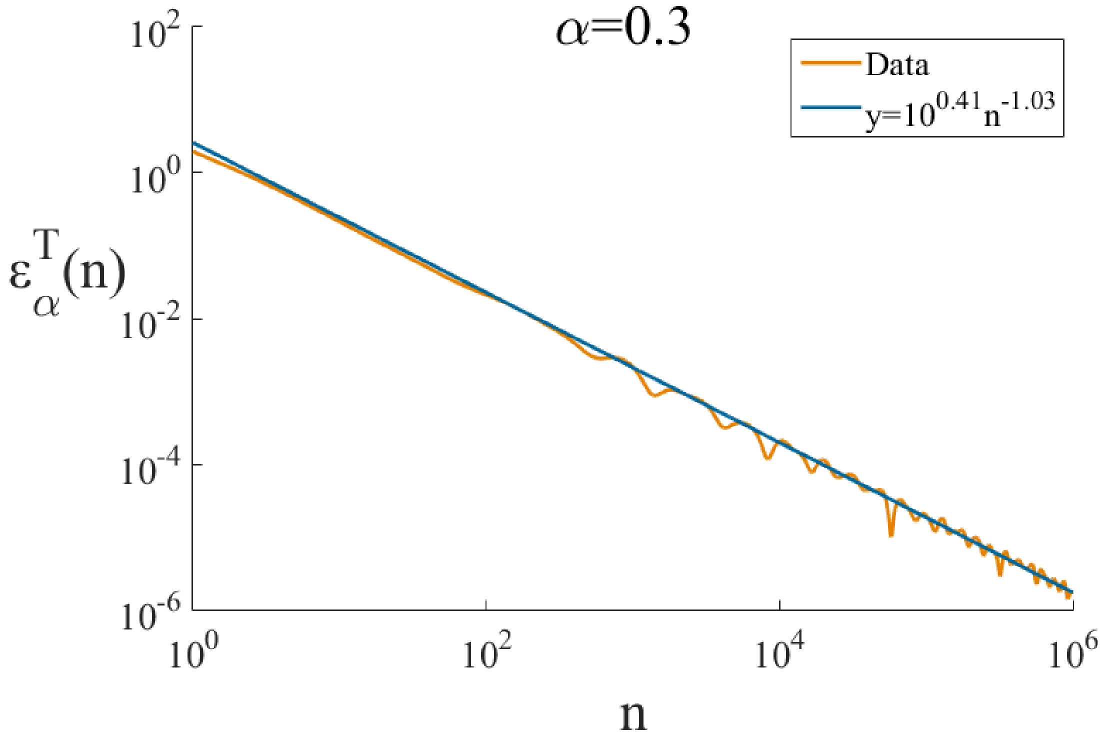

Figure 1: Numerical evaluation of the error in Eq. (6) with $K(n) = n^{0.3}$

valid in the large n limit as long as $K(n)$ grows sublinearly with n, i.e:

$$\lim_{n\to\infty} \frac{K(n)}{n} = 0. \tag{7}$$

As an example, Figure1 and Figure2 show the validity of (6) for the particular choice $K(n) = n^{\alpha}$ with $\alpha = 0.3$ and $\alpha = 0.8$ respectively. From the analytic point of view I have been able to prove that such a product formula works, even if not with the optimal bound suggested by the numerics. In order to obtain a bound on the error as good as possible I have used both the original Trotter approach and my analytic proof of the bang-bang decoupling, obtaining that:

$$\left(e^{-i\frac{t}{n}K(n)V}e^{-i\frac{t}{n}H}\right)^n = e^{-it(K(n)V+H)} + \min\left\{\mathcal{O}\left(\frac{K(n)}{n}\right), \mathcal{O}\left(\frac{1}{K(n)}\right)\right\}. \tag{8}$$

Resource theories and majorization

A resource theory is concerned about what kind of transformations can be done on a system when the set of allowed operations is restricted by some constraints. The states which can be obtained using these operations are called *free states*, whereas the other states represent a *resource*. The aim of a resource theory is to answer several questions about convertibility between these resource states, for example:

- When a deterministic interconversion between two resource states is possible?
- If we allow a certain probability of failure in the conversion, can we unlock some state transformations which are not feasible in a deterministic way?
- If we have at our disposal many copies of a given state, can we use them to convert them into several copies of another state in the asymptotic regime of infinitely many copies? If so, what is the optimal conversion rate we can achieve?

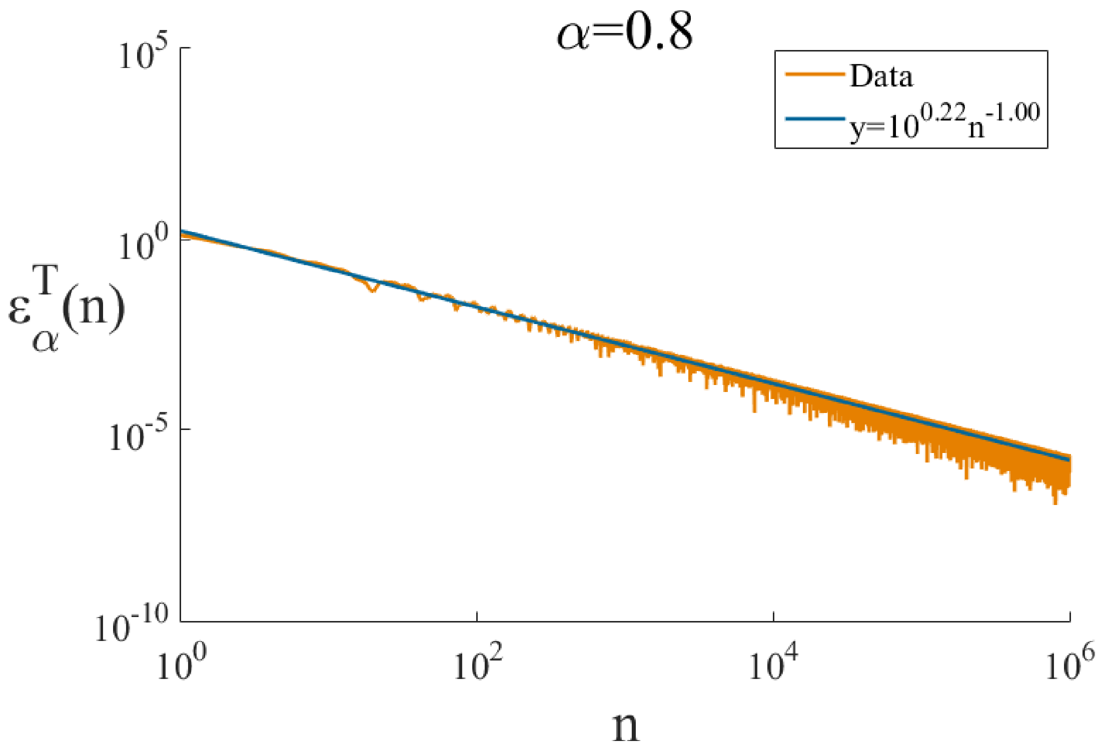

Figure 2: Numerical evaluation of the error in Eq. (6) with $K(n) = n^{0.8}$

- The availability of a catalyzer enable us to realize transformations which are not achievable otherwise?

Each of the previous questions constitutes a different aspect of a given resource theory.

Due to its very nature, thermodynamics can be cast as a resource theory when we precisely define what are the allowed operations which we can perform on a given system. For example, if we want to characterize the possible state transformations when our system is in contact with a thermal reservoir, we can attach as many Gibbs states at the temperature of the reservoir, and we can perform every unitaries which preserve the total energy of system and bath: this restrictions define the resource theory of thermal operations.

In my work, I consider the quantum resource theory of noisy operations, where the only operations we are allowed to do on the state ρ of a system are expressed by maps $\mathcal{E}$ of the kind:

$$\mathcal{E}(\rho) = \mathrm{Tr}_A \left[U \left(\rho \otimes \frac{\mathbb{I}_{\mathbb{A}}}{d} \right) U^\dagger \right], \tag{9}$$

namely, we can attach to our state an arbitrary system in the uniform state, we can perform every unitaries on the total system and we can eventually discard any subsystem. These restrictions allow us only to "inject noise" into the system, and reflect the fact that in nature everything evolves towards disorder. The resource theory of noisy operations corresponds to the resource theory of thermal operations when all Hamiltonians are trivial. The problem of convertibility in this resource theory translates into the problem of majorization: it can be shown that it is possible to convert ρ into σ if and only if their spectra $\lambda(\rho)$ and $\lambda(\sigma)$ are related by:

$$\lambda(\rho) \succ \lambda(\sigma), \tag{10}$$

where $\succ$ denotes the majorization relation. More precisely, given two probability vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ and denoted with $x_k^{\downarrow}$ the components ordered in decreasing order, $x \succ y$ means that:

$$\sum_{k=1}^{l} x_k^{\downarrow} \geqslant \sum_{k=1}^{l} y_k^{\downarrow} \qquad \forall l = 1, \ldots, n. \tag{11}$$

Majorization comes out also in the resource theory of entanglement, when one is concerned about the state conversions which are possible when we are allowed to perform only local operations and classical communications on a bipartite system, and entanglement is a resource since it cannot be created acting locally. In this context we have that a given state ψ of a bipartite system $\mathscr{H}_A \otimes \mathscr{H}_B$ can be converted into another state $\varphi \in \mathscr{H}_A \otimes \mathscr{H}_B$ if and only if $\lambda_A(\psi) \prec \lambda_A(\varphi)$, where $\lambda_A(\psi)$ denotes the entanglement spectrum of ψ, namely the spectrum of the reduced density matrix $\rho_A(\psi) = \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$. In the resource theory of entanglement we also have a criterion for determining the probability of a given state conversion when we allow some probability of failure (non-deterministic conversion). The probability of success in such interconversion is given by:

$$P(\psi \to \varphi) = \min_{l \in \{1, \ldots, n\}} \frac{\sum_{k=l}^{n} \lambda_A(\psi)}{\sum_{k=l}^{n} \lambda_A(\varphi)}, \tag{12}$$

which gives as a particular case deterministic conversion ($P(\psi \to \varphi) = 1$) if the majorization relation $\lambda_A(\psi) \prec \lambda_A(\varphi)$ holds.

We have studied numerically the statistical behaviour of such probability of conversion in the thermodynamic limit, when the dimension of $\mathscr{H}_A$ and $\mathscr{H}_B$ becomes very large. More precisely, we have considered what happens if we draw randomly two states $\psi, \varphi \in \mathscr{H}_A \otimes \mathscr{H}_B$ from the unitarily invariant distribution (Haar distribution) when $d_A = \dim \mathscr{H}_A$ and $d_B = \dim \mathscr{H}_B$ become large, while keeping constant their ratio $c = d_B/d_A$. Since the criteria we analyse are concerned only about the entanglement spectrum of the states, we have been able to use techniques from random matrix theory[9,10] to sample efficiently entanglement spectra corresponding to states drawn with the Haar measure over the total Hilbert space. This analysis enabled us to obtain several results, some of which are shown in Figure 3 and Figure 4.

These results are very interesting, since they tell us that when considering macroscopic systems it will be very unlikely that we can deterministically convert one state chosen randomly into another (it will be very unlikely that $P(\psi \to \varphi) = 1$), but if we allow for a certain probability of failure, this probability will tipically be small (and $\langle P(\psi \to \varphi) \rangle \simeq 1$) when $c > 1$.

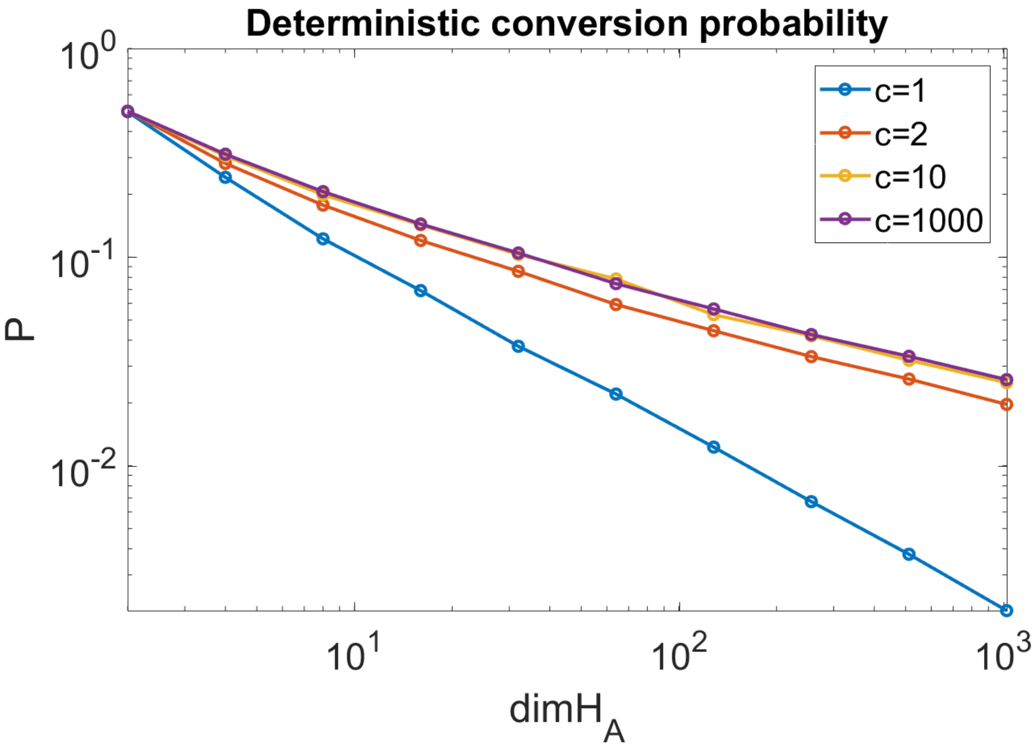

Figure 3: Probability to deterministically convert a state ψ to a state φ when the total state of $\mathscr{H}_A \otimes \mathscr{H}_B$ is drawn from the Haar distribution.

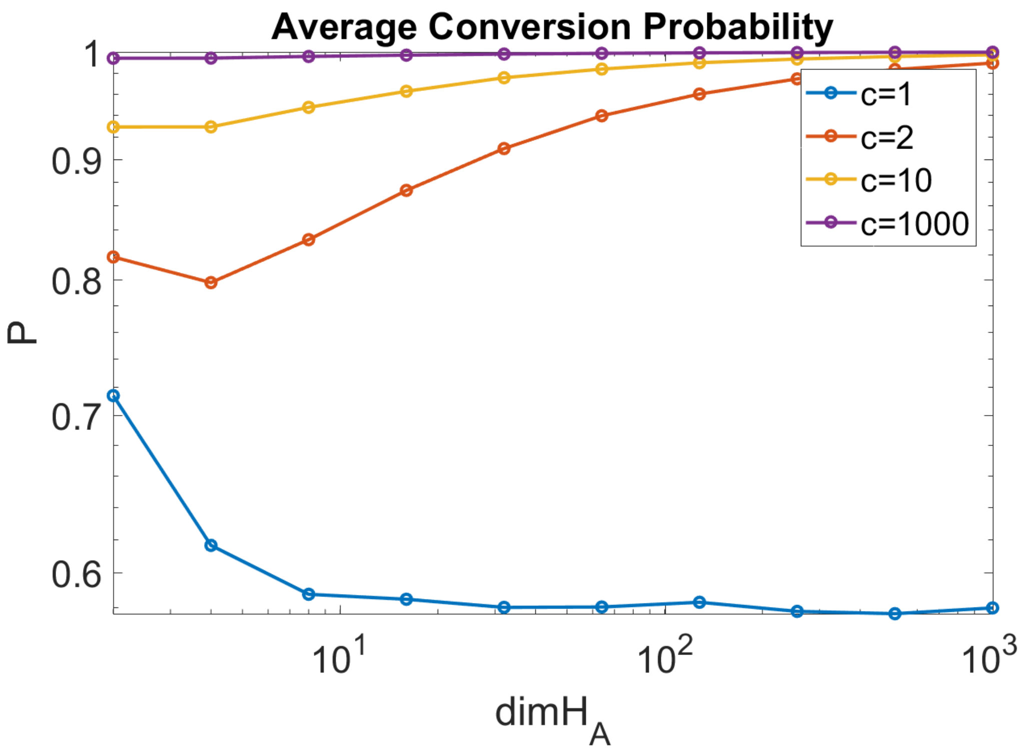

Figure 4: Average probability of (non-deterministic) conversion between two states randomly sampled from the Haar distribution.

Training activities

During the first year of my PhD I attended the following courses:

- Prof.ssa D'Orazio, *Management and knowledge of European research model and promotion of research result*s (completed);
- Prof.ssa White, *How to prepare a technical speech in English* (completed);
- Prof. Diacono, *Programing with Python for Data Science* (completed);
- Prof. Cafagna, *Introduction to C++ programming* (final exam to be taken);
- Prof. Gonnella, *Linear Stability analysis* (course still in progress to date);
- Prof. Pascazio, *Differential equations and physical phenomena* (final exam to be taken);
- Prof. Facchi, *States, observables and Evolution* (completed);
- Prof. Pepe, *Atom-photon interactions* (completed).

Schools and Conferences

I attended the following scientific workshops and conferences:

- 13–15 December 2017, *Statistical Mechanics & Field Theory*, Bari;
- 19–24 February 2018, *Mathematical Challenges in Quantum Mechanics*, Rome;
- 24–25 March 2018, *Current Problems in Theoretical Physics*, Vietri sul Mare (SA);
- 5–12 June 2018, *Seminario Nazionale di Fisica Nucleare e Subnucleare "Francesco Romano"*, Otranto (LE);
- 25–29 June 2018, *Information Geometry, Quantum Mechanics and Applications*, San Rufo (SA), with the presentation of the talk "Continuous and Pulsed Quantum Control";
- 17–20 September 2018, *Italian Quantum Information Science conference*, Catania, with the presentation of the poster "Continuous and Pulsed Quantum Control".

References

[1] C. H. Bennett,H. J. Bernstein, S. Popescu, B. Schumacher, *Concentrating partial entanglement by local operations*, Physical Review A **53**, 2046 (1996).

[2] F. G. Brandao, M. Horodecki, J. Oppenheim, J. M. Renes, R. W. Spekkens, *Resource theory of quantum states out of thermal equilibrium*, Physical Review Letters **111**, 250404 (2013).

[3] C. Sparaciari, J. Oppenheim, T. Fritz, *Resource theory for work and heat*. Physical Review A **96**, 052112 (2017).

[4] Goold J., Huber M., Riera A., del Rio L., Skrzypczyk P., *The role of quantum information in thermodynamics—a topical review*, Journal of Physics A: Mathematical and Theoretical **49**, 143001 (2016)

[5] M. Horodecki, J. Oppenheim, *Fundamental limitations for quantum and nanoscale thermodynamics*, Nature communications **4**, 2059 (2013).

[6] L. Viola, E. Knill, E., S. Lloyd, *Dynamical decoupling of open quantum systems*, Physical Review Letters **82**, 2417 (1999).

[7] D. Burgarth, P. Facchi, G. Gramegna, S. Pascazio, *Continuous and pulsed quantum control* (in preparation).

[8] F.D. Cunden, P. Facchi, G. Florio, G. Gramegna, *Majorization in high dimensional spaces* (in preparation).

[9] A. Edelman, B.D. Sutton, Y. Wang, *Random Matrix Theory, Numerical Computation and Applications* Modern Aspects of Random Matrix Theory **72**, 53 (2012).

[10] A. Edelman, Y. Wang *Random Matrix Theory and its Innovative Applications* Advances in Applied Mathematics, Modeling, and Computational Science (pp. 91-116). Springer, Boston, MA (2013).