Rebirth and golden age of chaos theory

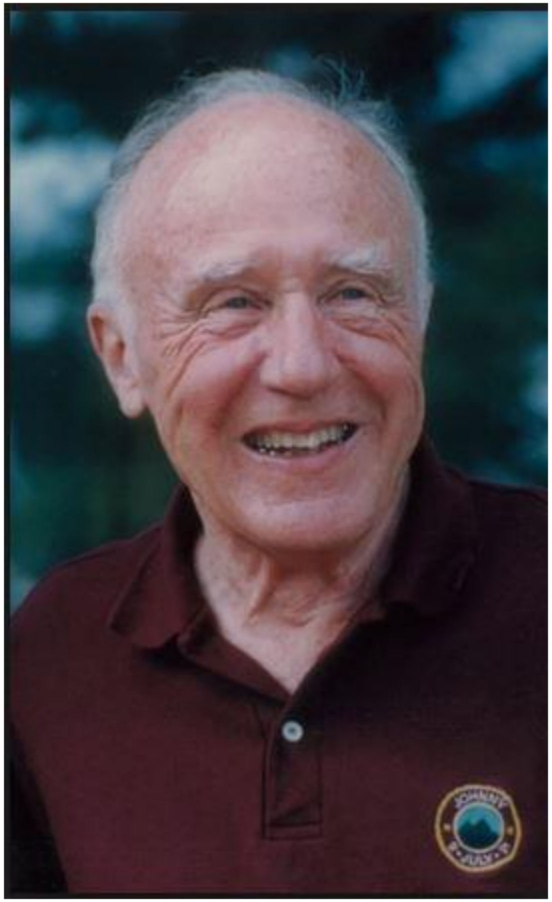
- Lorenz and butterfly effect and attractors
- Ruelle (mathematical formalism) and strange attractors (chaotic system representation in a particular phase space)
- types of attractors (fixed point, limit-cycle, limit-thorus, strange attractor)
- Feigenbaum (periodic doubling, logistic map)
- Mandelbrot (roughness theory, fractals)
- Thom (catastrophe theory, "prevoir n'est pas expliquer")
- Prigogine (theory of dissipative structures)
- examples

Edward Lorenz, 1963 ("Predictability; Does the Flap of a Butterfly's wings in Brazil Set Off a Tornado in Texas?"(1972))

«Let I appear frivolous in even posing the title question, let alone suggesting that it might have an affirmative answer, let me try to place it in proper perspective by offering two propositions.

1. If a single flap of a, not to mention the activities of innumerable more powerful creatures, including butterfly's wings can be instrumental in generating a tornado, so also can all the previous and subsequent flaps of its wings, as can the flaps of the wings of millions of other butterflies our own species.
2. If the flap of butterfly's wings can be instrumental in generating a tornado, it can well be instrumental in preventing a tornado. …if two particular weather situations differing by as little as the immediate influence of a single butterfly will generally after sufficient time evolve into two situations differing by as much as the presence of a tornado. In more technical language, is the behavior of the atmosphere unstable with respect to perturbations of a small amplitude?

   The connection between this question and our ability to predict the weather is evident. Since we not know exactly how many butterflies there are, nor where they are all located, let alone which ones are flapping their wings at any instant, we cannot, if the answer to our question is affirmative, accurately predict the occurrence of tornados at a sufficient distant future time»

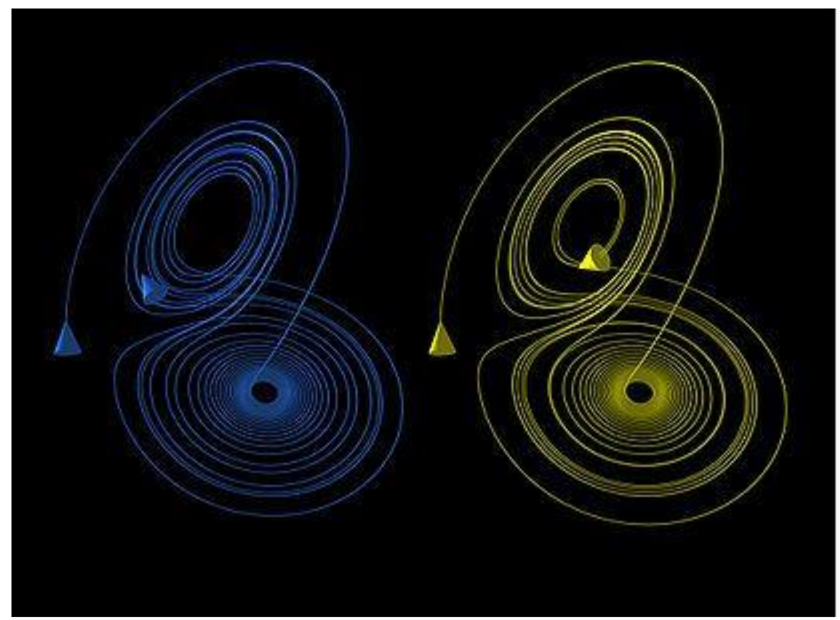


Edward Lorenz (1917-2008)

Lorenz’ equations

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - xz - y \\ \dot{z} = xy - \beta z \end{cases}$$

where σ=Prandtl’ number f(viscosity); ρ=Rayleigh’ number (control parameter f(T)); β=parameter f(geometry); often σ=10, β=8/3, ρ=28

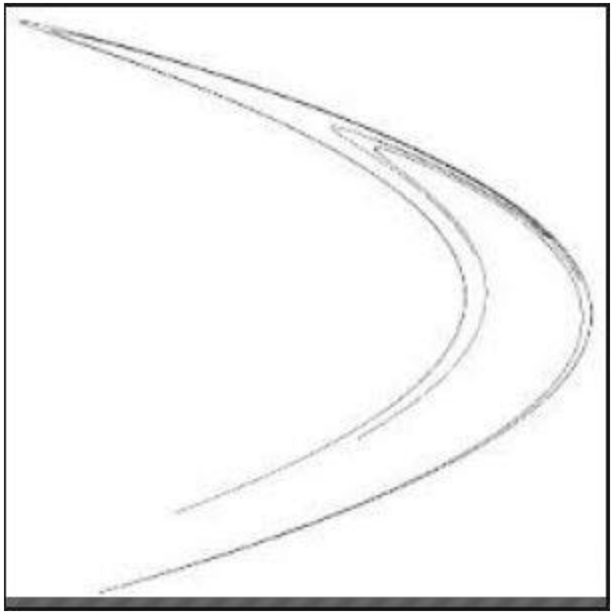


z(t)-z(t)=10^{-5}

Michel Hénon (1931-2013)

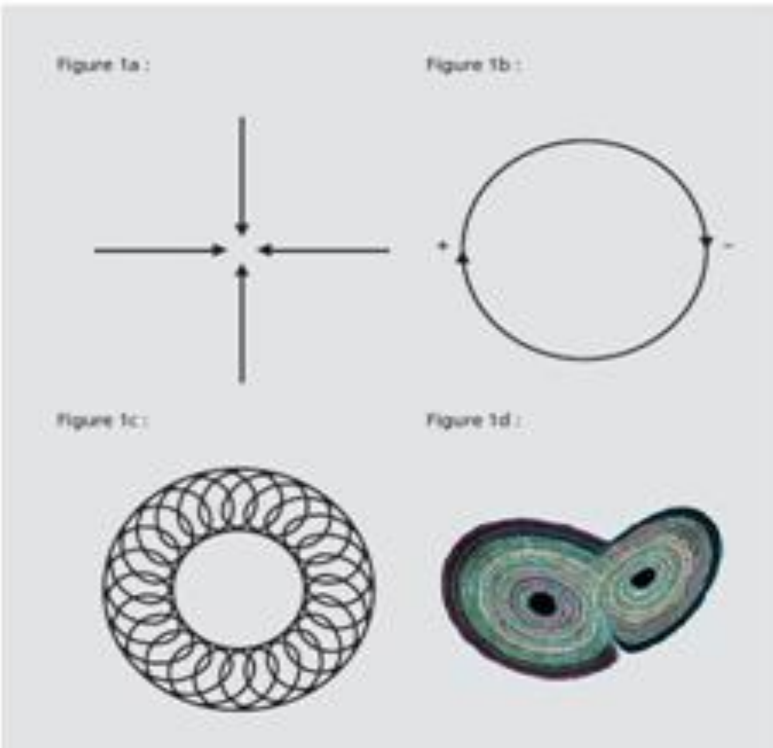
Attrattore di Hénon

$$\begin{cases} x_{t+1} = y_t + 1 - 1.4x_t^2 \\ y_{t+1} = 0.3x_t \end{cases}$$



David Ruelle, 1971 introduces the name strange attractor

The strange attractor is a representation of a chaotic system in a specific phase space, but attractors are found in many dynamical systems that are non-chaotic. There are four types of attractors: fixed point, limit-cycle, limit-torus, and strange attractor.



Feigenbaum and the logistic map, 1978

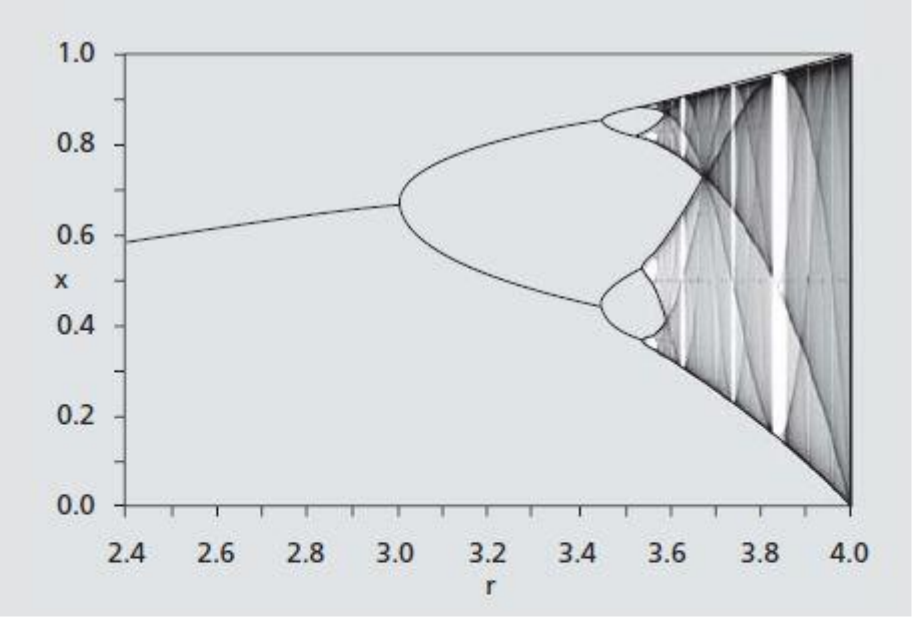
The logistic map is a function of the segment [0,1] within itself defined by:

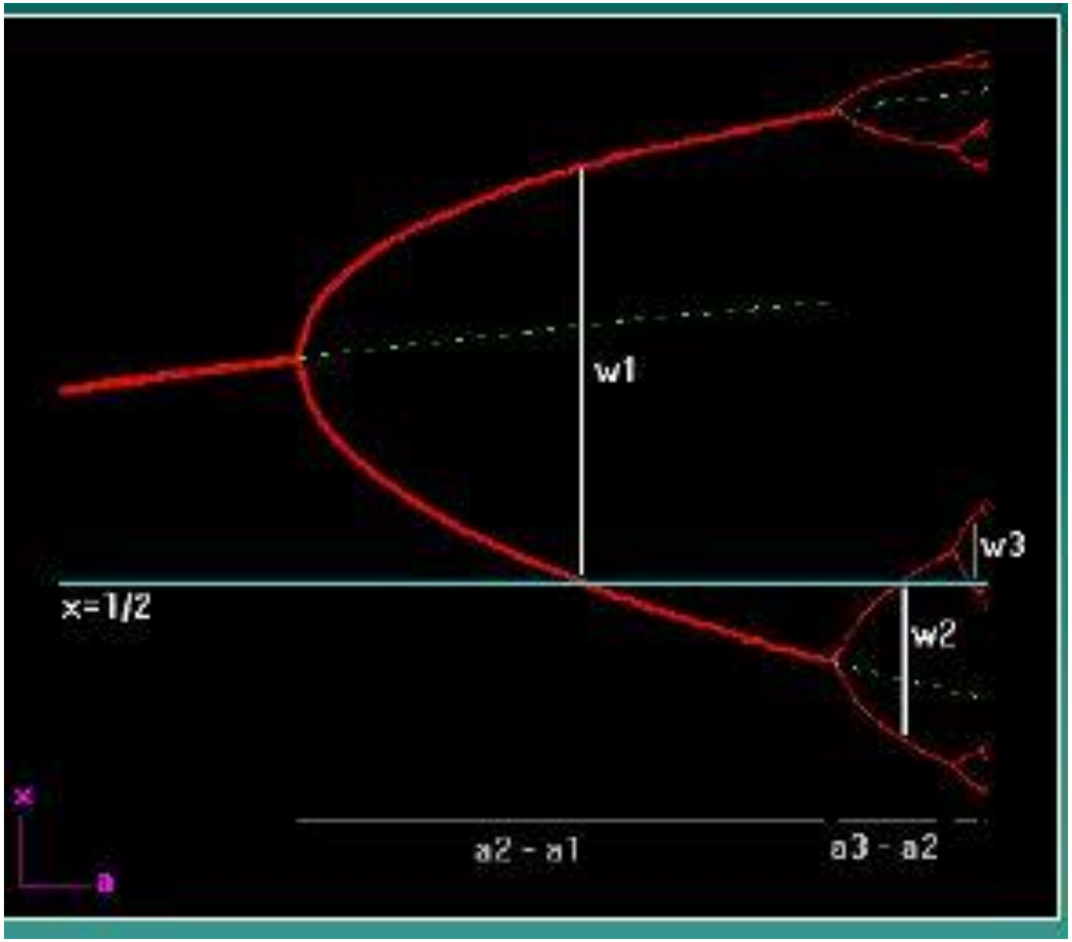
$$x_{n+1} = rx_n(1- x_n)$$

where n = 0, 1, … describes the discrete time, the single dynamical variable, and 0≤r≤4 is a parameter. The dynamic of this function presents very different behaviors depending on the value of the parameter r: for 0≤ r <3, the system has a fixed point attractor that becomes unstable when r=3. For 3≤ r <3,57…, the function has a periodic orbit as attractor, of a period of 2n where n is an integer that tends towards infinity when r tends towards 3,57… When r=3,57..., the function then has a Feigenbaum fractal attractor. When over the value of r=4, the function goes out of the interval [0,1].

Logistic map was introduced by the biologist Robert M. May, 1976

(natural bifurcations: periodic (one point); period doubling (two points); chaotic system (multiple points))





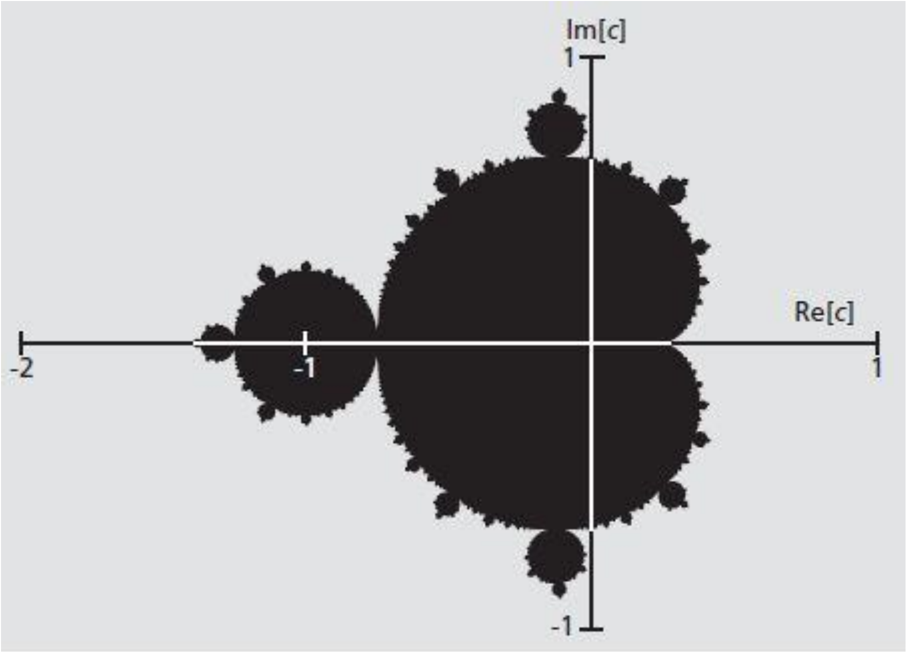
The order-chaos transition follows very precise universal laws

Several natural multiple bifurcations are characterized by constants discovered by Feigenbaum:

$$\delta = \lim_{n\to\infty} (a_n - a_{n+1})/(a_{n+1} - a_n) = 4.6692016091$$

$$\alpha = \lim_{n\to\infty} w_n / w_{n+1} = 2.5029078750$$

Mandelbrot and fractal dimensions,1973



«The basic concept that combines the study of fractals to the disciplines such as biology and anatomy and physiology is the conviction of a necessary overcoming of Euclidean geometry in the description of natural reality. Wanting to be very synthetic, fractals are used to find a new representation which starts from the basic idea that the small in nature is nothing but a copy of the great. My belief is that fractals will soon be employed in understanding the neural processes , the human mind will be their new frontier.»

Benoit Mandelbrot (1924-2010)

fractus→dimension≠ integer

$$z_{n+1} = z_n^{\,2} + c$$

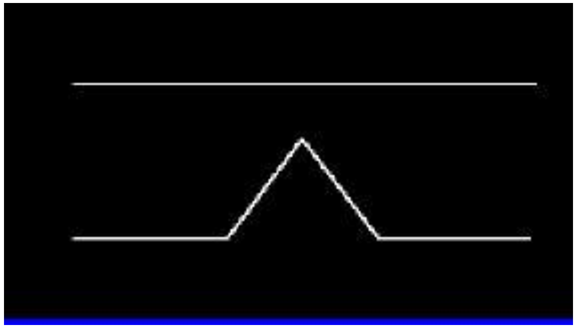
con $z_0 = 0$

Fractal Dimension: some examples
(from a Husdorff's definition, 1919

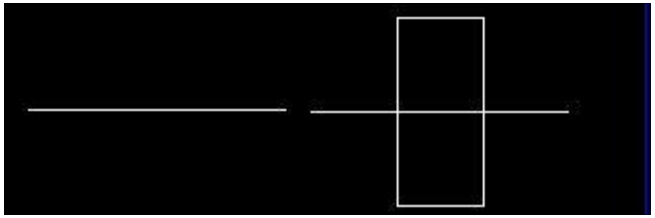
n= number of linear magnifications;
f(n)=number of object copies
d=fractal dimension

$\longrightarrow f(n)=n^d \longrightarrow$ $d=\log[f(n)]/\log n$

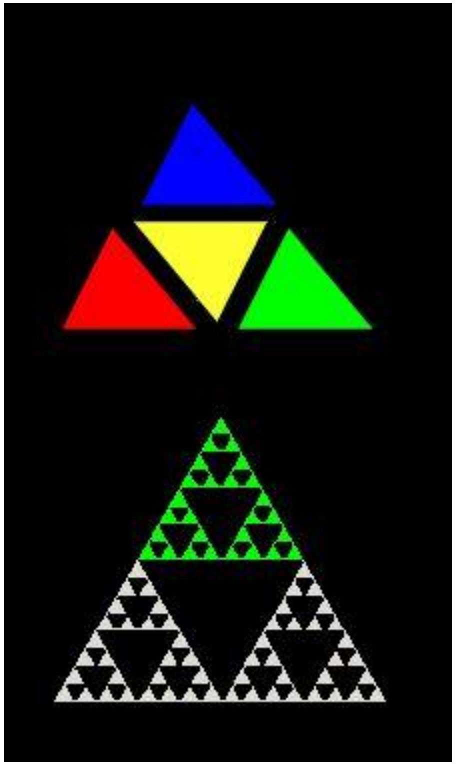
Cantor d=log(2)/log (3)=0.63



Koch d=log(4)/log (3)=1.26



Peano d=log(9)/log (3)=2 (surface)



Sierpinski

$d=\log(3)/\log(2)=1,5849625$

Features and classification

(a) physical-geometrical
(b) analytical-geometrical
(c) dynamical-physical

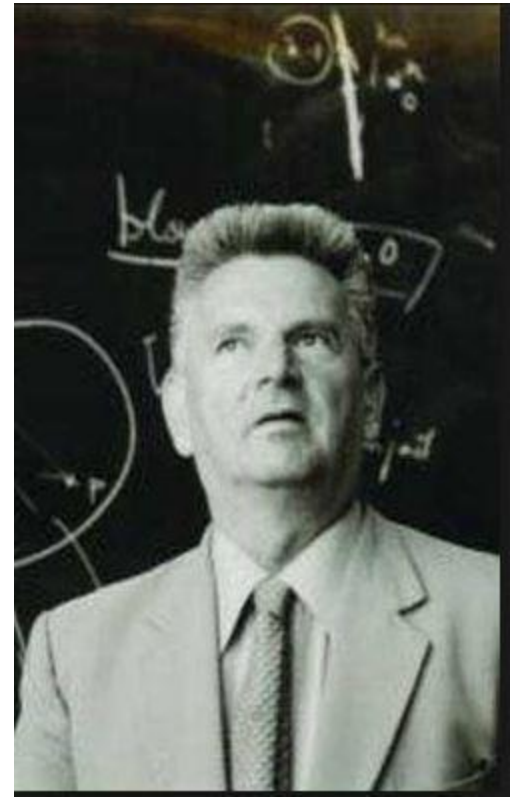
- ►self-similarity
- ►self-affinity

F is a fractal if its dimension is strictly higher than its topological dimension

The fractal dimension is linked to the way in which the object fills the space in which it is contained , to its roughness and contains a lot of information on the geometric properties of the object

A large number of phenomena can be described essentially by means of non-linear differential equations … Far from equilibrium, it can create coherent states and complex structures that could not exist in a reversible world. This depends on a fundamental property of the dissipative phenomena that mechanical systems do not possess: the asymptotic stability. [...] In dissipative systems it is possible to forget the perturbations and differences in initial conditions.

Ilya Prigogine (1917-2003)



René Thom (1923-2002)

Modèles mathématiques de la morphogénèse,

Paris: C. Bourgois, 1974

Prédire n'est pas expliquer, Paris: Eshel, 1991

Study of singularities: continuous actions produce erratic results

Table I. Definitions of concepts related to the history of chaos theory.*

- **Causality principle**. Every effect has an antecedent, proximate cause.
- **Determinism**. A philosophical proposition that every event is physically determined by an unbroken chain of prior occurrences.
- **Predictability**. This refers to the degree that a correct forecast of a system's state can be made either qualitatively or quantitatively.
- **Model**. A pattern, plan, representation, or description designed to show the structure or workings of an object, system, or concept.
- **Dynamical system**. A system that changes over time in both a causal and a deterministic manner, ie, its future depends only on phenomena from its past and its present (causality) and each given initial condition will lead to only one given later state of the system (determinism). Systems that are noisy or stochastic, in the sense of showing randomness, are not dynamical systems, and the probability theory is the one to apply to their analysis.
- **Phase space**. An abstract space in which all possible states of a system are represented which, each possible state of the system corresponding to one unique point in the phase space.
- **Sensitivity to initial conditions**. This is when a change in one variable has the consequence of an exponential change in the system.
- **Integrable system**. In mathematics, this refers to a system of differential equations for which solutions can be found. In mechanics, this refer to a system that is quasiperiodic.
- **Linear system**. A system is said to be linear when the whole is exactly equal to the sum of its components.
- **Attractor**. A set to which a dynamical system evolves after a long enough time.

- **Characteristic Lyapunov time**. The characteristic time of a system is defined as the delay When changes from the initial point are multiplied by 10 in the phase space.
- **Feedback**. A response to information, that either increases effects (positive feedback), or decreases them (negative feedback), or induces a cyclic phenomenon.
- **Self-similarity**. This means that an object is composed of subunits and sub-subunits on multiple levels that (statistically) resemble the structure of the whole object. However, in every day life, there are necessarily lower and upper boundaries over which such self-similar behavior applies.
- **Fractal**. Is a geometrical object satisfying two criteria: self-similarity and fractional dimensionality.
- **Fractal dimension**. Let an object in a n-dimensions space be covered by the smallest number of open spheres of radius r. The fractal dimension is log(N)/log(1/r) when r tends towards 0.

(*) Oestreicher, 2007

From Chaos to Complex Systems

After simplicity problems and non-organized complexity problems, chaotic phenomena, here organized complexity problems (Weaver). There is an enormous difference between complexity and chaos. It is crucial to discover differences and analogies. Before giving as complete as possible definition of what is a complex system, on which most researchers agree, it may be more useful to ask if the system I'm studying is a complex system. Is it possible the second question without answering the first? What about the role of composition and connection between its parts?

Let we examine some example and some features

At beginning of XX century, quantum physics comes from having done insistently classical Newtonian questions to systems that did not respond in classical way, and then science has invented a new way to look at these phenomena...

After a big development of quantum physics, the other great moment of conceptual expansion was precisely the study of complex systems: superfluids, meteorology, phase transitions, living systems, crystals and cognitive processes.

It is the mesorealm; the middle region (Weaver); the middle way, as referred by the Physics Nobel Prize (1998 - quantum Hall effect) Robert Laughlin (1950 - …).

The crucial feature is the interplay of dynamics and scales to describe interesting facts typical of middle way.

In this mesorealm is not too much important the number of ariable: the really important characteristic lies in the fact that problems show the essential feature of organization.

Problem of organized complexity: constant relationships between system and environment, with very different and intricate spatial and temporal distribution.

«What make an evening primrose open when it does? Why does salt water fail to satisfy thirst? Why is one chemical substances a poison when another, whose molecules have just the same atoms but assembled into mirror-image pattern, is completely harmless? What is description of age in biological terms? Is virus a living organism? Does complex protein "know how" to reduplicate their pattern?» [Weaver]

For all these problems, statistical methods fail: *they involve dealing simultaneously with a sizable number of factors which are interrelated into an organic whole: the organized complexity*.

We are ready for definition of features of a complex system and its evolution (the identikit)

- it is typically open (matter, energy, information fluxes): all real system are open
- it contains a set of many interacting objects, or “agents” (physical interaction, group interaction, information exchange, …, networks): generally non-linear interactions, delayed effects
- repetitive (gas, water) and non-repetitive (nervous central systems, neurons, cellular enzyme ensemble)
- the behavior is conditioned by memory (negative and positive feedback cycles)
- it presents stability and robustness (perturbation damping) and differentiated sensibility (presence of critical points relative to external injury)
- it is adaptive (CAS): agents can modify their strategies
- hierarchical organization (each sub-system is also complex): an organized system is able to evolve more faster
- partial agents autonomy
- presence of paradoxes

What about evolution

- it appears like "living " system (its behavior is absolutely non-trivial)
- it gives rise to emergent phenomena, often astonishing and sometime extreme (creativity, innovation, unpredictability, no control)
- auto-organization (no external intervention)
- it alternates ordered and chaotic behavior in a complicated way
- simple agents interconnected can produce a large spectrum of realistic results: this is the complexity essence, sometime a slap to the reductionist approach to the knowledge of the world.

Therefore, we can say that the key difference between a generic system, more o less complicated, and a complex system is in the fact that the relationship between its parts is more important than part constitution

Throughout the natural and artificial world one observes phenomena of the great complexity. Yet research in physics and to some extent biology and other fields has shown that the basic components of many systems are quite simple. It is now a crucial problem for many areas of science to elucidate the mathematical mechanisms by which large numbers of such simple components, acting together, can produce behaviour of the great complexity observed. One hopes that It will be possible to formulate universal laws that escribe such complexity. [Wolfram, 1985]

Stephen Wolfram (1959 - …)